

# Sums of read-once formulas: How many summands suffice?

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**Abstract.** An arithmetic read-once formula (ROF) is a formula (circuit of fan-out 1) over  $+$ ,  $\times$  where each variable labels at most one leaf. Every multilinear polynomial can be expressed as the sum of ROFs. In this work, we prove, for certain multilinear polynomials, a tight lower bound on the number of summands in such an expression.

## 1 Introduction

Read-once formulae (ROF) are formulae (circuits of fan-out 1) in which each variable appears at most once. A formula computing a polynomial that depends on all its variables must read each variable at least once. Therefore, ROFs compute some of the simplest possible functions that depend on all of their variables. The polynomials computed by such formulas are known as read-once polynomials (ROPs). Since every variable is read at most once, ROPs are multilinear<sup>1</sup>. But not every multilinear polynomial is a ROP. For example,  $x_1x_2 + x_2x_3 + x_1x_3$ .

We investigate the following question: Given an  $n$ -variate multilinear polynomial, can it be expressed as a sum of at most  $k$  ROPs? It is easy to see that every bivariate multilinear polynomial is an ROP. Any tri-variate multilinear polynomial can be expressed as a sum of 2 ROPs. With a little thought, we can obtain a sum-of-3-ROPs expression for any 4-variate multilinear polynomial. An easy induction on  $n$  then shows that any  $n$ -variate multilinear polynomial, for  $n \geq 4$ , can be written as a sum of at most  $3 \times 2^{n-4}$  ROPs. We ask the following question: Does there exist a strict hierarchy among  $k$ -sums of ROPs? Formally,

*Problem 1.* Consider the family of  $n$ -variate multilinear polynomials. For  $1 < k \leq 3 \times 2^{n-4}$ , is  $\sum^k \cdot \text{ROP}$  strictly more powerful than  $\sum^{k-1} \cdot \text{ROP}$ ? If so, what explicit polynomials witness the separations?

We answer this affirmatively for  $k \leq \lceil n/2 \rceil$ . In particular, for  $k = \lceil n/2 \rceil$ , there exists an explicit  $n$ -variate multilinear polynomial which cannot be written as a sum of less than  $k$  ROPs but it admits a sum-of- $k$ -ROPs representation.

Note that  $n$ -variate ROPs are computed by linear sized formulas. Thus if an  $n$ -variate polynomial  $p$  is in  $\sum^k \cdot \text{ROP}$ , then  $p$  is computed by a formula of size

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<sup>1</sup> A polynomial is said to be multilinear if the individual degree of each variable is at most one.

$O(kn)$  where every intermediate node computes a multilinear polynomial. Since superpolynomial lower bounds are already known for the model of multilinear formulas [7], we know that for those polynomials (including the determinant and the permanent), a  $\sum^k \cdot \text{ROP}$  expression must have  $k$  at least quasi-polynomial in  $n$ . However the best upper bound on  $k$  is only exponential in  $n$ , leaving a big gap between the lower and upper bound. On the other hand, our lower bound is provably tight.

A natural question to ask is whether stronger lower bounds than the above result can be proven. In particular, to separate  $\sum^{k-1} \cdot \text{ROP}$  from  $\sum^k \cdot \text{ROP}$ , how many variables are needed? The above hierarchy result says that  $2k - 1$  variables suffice, but there may be simpler polynomials (with fewer variables) witnessing this separation. We demonstrate another technique which improves upon the previous result for  $k = 3$ , showing that 4 variables suffice. In particular, we show that over the field of reals, there exists an explicit multilinear 4-variate multilinear polynomial which cannot be written as a sum of 2 ROPs. This lower bound is again tight, as there is a sum of 3 ROPs representation for every 4-variate multilinear polynomial.

## Our results and techniques

We now formally state our results.

**Theorem 1.** *For each  $n \geq 1$ , the  $n$ -variate degree  $n - 1$  symmetric polynomial  $S_n^{n-1}$  cannot be written as a sum of less than  $\lceil n/2 \rceil$  ROPs, but it can be written as a sum of  $\lceil n/2 \rceil$  ROPs.*

The idea behind the lower bound is that if  $g$  can be expressed as a sum of less than  $\lceil n/2 \rceil$  ROFs, then one of the ROFs can be eliminated by taking partial derivative with respect to one variable and substituting another by a field constant. We then use the inductive hypothesis to arrive at a contradiction. This approach necessitates a stronger hypothesis than the statement of the theorem, and we prove this stronger statement in Theorem 6.

**Theorem 2.** *There is an explicit 4-variate multilinear polynomial  $f$  which cannot be written as the sum of 2 ROPs over  $\mathbb{R}$ .*

The proof of this theorem mainly relies on a structural lemma (Lemma 6) for sum of 2 read-once formulas. In particular, we show that if  $f$  can be written as a sum of 2 ROPs then one of the following must be true: 1. Some 2-variate restriction is a linear polynomial. 2. There exist variables  $x_i, x_j \in \text{Var}(f)$  such that the polynomials  $x_i, x_j, \partial_{x_i}(f), \partial_{x_j}(f)$  are affinely dependent. 3. We can represent  $f$  as  $f = l_1 \cdot l_2 + l_3 \cdot l_4$  where  $(l_1, l_2)$  and  $(l_3, l_4)$  are variable-disjoint linear forms. Checking the first two conditions is easy. For the third condition we use the commutator of  $f$ , introduced in [10], to find one of the  $l_i$ 's. The knowledge of one of the  $l_i$ 's suffices to determine all the linear forms. Finally, we construct a 4-variate polynomial which does not satisfy any of the above mentioned conditions. This construction does not work over algebraically closed fields. We do not yet know

how to construct an explicit 4-variate multilinear polynomial not expressible as the sum of 2 ROPs over such fields, or even whether such polynomials exist.

## Related work

Despite their simplicity, ROFs have received a lot of attention both in the arithmetic as well as in the Boolean world [4,3,1,2,8,10]. The most fundamental question that can be asked about polynomials is the polynomial identity testing (PIT) problem: Given an arithmetic circuit  $\mathcal{C}$ , is the polynomial computed by  $\mathcal{C}$  identically zero or not. PIT has a randomized polynomial time algorithm: Evaluate the polynomial at random points. It is not known whether PIT has a deterministic polynomial time algorithm. In 2004, Kabanets and Impagliazzo established a connection between PIT algorithms and proving general circuit lower bounds [5]. However, for restricted arithmetic circuits, no such result is known. For instance, consider the case of multilinear formulas. Even though strong lower bounds are known for this model, there is no efficient deterministic PIT algorithm. For this reason, PIT was studied for the weaker model of sum of read-once formulas. Notice that multilinear depth 3 circuits are a special case of this model.

Shpilka and Volkovich gave a deterministic PIT algorithm for the sum of a small number of ROPs [8]. Interestingly, their proof uses a lower bound for a weaker model, that of 0-justified ROFs. In particular, they show that the polynomial  $\mathcal{M}_n = x_1 x_2 \cdots x_n$ , consisting of just a single monomial, cannot be represented as a sum of less than  $n/3$  weakly justified ROPs. More recently, Kayal showed that if  $\mathcal{M}_n$  is represented as a sum of powers of low degree (at most  $d$ ) polynomials, then the number of summands is at most  $\exp(\Omega(n/d))$  [6]. He used this lower bound to give a PIT algorithm. Our lower bound from Theorem 1 is orthogonal to both these results and is provably tight. An interesting question is whether it can be used to give a PIT algorithm.

### 1.1 Organization

The paper is organized as follows. In Section 2 we give the basic definitions and notations. In Section 3, we establish Theorem 1, showing that the hierarchy of  $k$ -sums of ROPs is proper. In Section 4 we establish Theorem 2, showing an explicit 4-variate multilinear polynomial that is not expressible as the sum of two ROPs. We conclude in Section 5 with some further questions that are still open.

## 2 Preliminaries

For a positive integer  $n$ , we denote  $[n] = \{1, 2, \dots, n\}$ . For a polynomial  $f$ , by  $\text{Var}(f)$  we mean the set of variables occurring in  $f$ . For a polynomial  $f(x_1, x_2, \dots, x_n)$ , a variable  $x_i$  and a field element  $\alpha$ , we denote by  $f|_{x_i=\alpha}$  the polynomial resulting from setting  $x_i = \alpha$ . Let  $f$  be an  $n$ -variate polynomial. We say that  $g$  is a

$k$ -variate restriction of  $f$  if  $g$  is obtained by setting some variables in  $f$  to field constants and  $|\text{Var}(g)| \leq k$ . A set of polynomials  $f_1, f_2, \dots, f_k$  over the field  $\mathbb{F}$  is said to be affinely dependent if there exist constants  $\alpha_1, \alpha_2, \dots, \alpha_k$  such that  $\sum_{i \in [k]} \alpha_i f_i = 0$ .

The  $n$ -variate degree  $k$  elementary symmetric polynomial, denoted  $S_n^k$ , is defined as follows:

$$S_n^k(x_1, \dots, x_n) = \sum_{A \subseteq [n], |A|=k} \prod_{i \in A} x_i.$$

A circuit is a directed acyclic graph with variables and field constants labeling the leaves, field operations  $+$ ,  $\times$  labeling internal nodes, and a designated sink node. Each node naturally computes a polynomial; the polynomial at the designated sink node is the polynomial computed by the circuit. If the underlying undirected graph is also acyclic, then the circuit is called a formula. A formula is said to be read- $k$  if each variable appears as a leaf label at most  $k$  times.

For read-once formulas, it is more convenient to use the following “normal form” from [8].

**Definition 1 (Read-once formulas [8]).** *A read-once arithmetic formula (ROF) over a field  $\mathbb{F}$  in the variables  $\{x_1, x_2, \dots, x_n\}$  is a binary tree as follows. The leaves are labeled by variables and internal nodes by  $\{+, \times\}$ . In addition, every node is labeled by a pair of field elements  $(\alpha, \beta) \in \mathbb{F}^2$ . Each input variable labels at most once leaf. The computation is performed in the following way. A leaf labeled by  $x_i$  and  $(\alpha, \beta)$  computes  $\alpha x_i + \beta$ . If a node  $v$  is labeled by  $\star \in \{+, \times\}$  and  $(\alpha, \beta)$  and its children compute the polynomials  $f_1$  and  $f_2$ , then  $v$  computes  $\alpha(f_1 \star f_2) + \beta$ .*

We say that  $f$  is a read-once polynomial (ROP) if it can be computed by a ROF, and is in  $\sum^k$ -ROP if it can be expressed as the sum of at most  $k$  ROPs.

**Proposition 1.** *For every  $n$ , every  $n$ -variate multilinear polynomial can be written as the sum of at most  $\lceil 3 \times 2^{n-4} \rceil$  ROPs.*

*Proof.* For  $n = 1, 2, 3$  this is easy to see.

For  $n = 4$ , let  $f(X)$  be given by the expression  $\sum_{S \subseteq [4]} A_S x_S$ , where  $x_S$  denotes the monomial  $\prod_{i \in S} x_i$ . We want to express  $f$  as  $f_1 + f_2 + f_3$ , where each  $f_i$  is an ROP. If there are no degree 2 terms, we use the following:

$$\begin{aligned} f_1 &= A_\emptyset + A_1 x_1 + A_2 x_2 + A_3 x_3 + A_4 x_4 \\ f_2 &= x_1 x_2 (A_{123} x_3 + A_{124} x_4) \\ f_3 &= x_3 x_4 (A_{134} x_1 + A_{234} x_2 + A_{1234} x_1 x_2) \end{aligned}$$

Otherwise, assume wlog that  $A_{13} \neq 0$ . Then define

$$\begin{aligned} f_1 &= \left[ \sum_{S \subseteq [2]} A_S \prod_{i \in S} x_i \right] + \left[ \sum_{\emptyset \neq S \subseteq \{3,4\}} A_S \prod_{i \in S} x_i \right] \\ f_2 &= (A_{13}x_1 + A_{23}x_2 + A_{123}x_1x_2) \cdot \left( \frac{A_{14}}{A_{13}}x_4 + x_3 + \frac{A_{134}}{A_{13}}x_3x_4 \right) \\ f_3 &= x_2x_4 \left[ \left( A_{24} - \frac{A_{14}A_{23}}{A_{13}} \right) + x_1 \left( A_{124} - \frac{A_{14}A_{123}}{A_{13}} \right) \right. \\ &\quad \left. + x_3 \left( A_{234} - \frac{A_{134}A_{23}}{A_{13}} \right) + x_1x_3 \left( A_{1234} - \frac{A_{134}A_{123}}{A_{13}} \right) \right] \end{aligned}$$

Since any bivariate multilinear polynomial is a ROP, each  $f_i$  is indeed an ROP.

For  $n > 4$ , express  $f$  as  $x_ng + h$  where  $g = \partial_{x_n} f$  and  $h = f|_{x_n=0}$ , and use induction, along with the fact that  $g$  does not have variable  $x_n$ .  $\square$

**Fact 3 (Useful Fact about ROPs [9])** *The partial derivatives of ROPs are also ROPs.*

**Proposition 2 (3-variate ROPs).** *Let  $f \in \mathbb{F}[x_1, x_2, x_3]$  be a 3-variate ROP. Then there exists  $i \in [3]$  and  $A \in \mathbb{F}$  such that  $\deg(f|_{x_i=A}) \leq 1$ .*

*Proof.* Assume without loss of generality that  $f = f_1(x_1) \star f_2(x_2, x_3) + c$  where  $\star \in \{+, \times\}$  and  $c \in \mathbb{F}$ . If  $\star = +$ , then for all  $A \in \mathbb{F}$ ,  $\deg(f|_{x_2=A}) \leq 1$ . If  $\star = \times$ ,  $\deg(f|_{f_1=0}) \leq 1$ .  $\square$

We will also be dealing with a special case of ROFs called multiplicative ROFs defined below:

**Definition 2 (Multiplicative Read-once formulas).** *A ROF is said to be a multiplicative ROF if it does not contain any addition gates. We say that  $f$  is a multiplicative ROP if it can be computed by a multiplicative ROF.*

Multiplicative ROPs have the following useful property, observed in [9]. (See Lemma 5.13 in [9]. For completeness, and since we refer to the proof later, we include a proof sketch here.)

**Lemma 1 ([9]).** *Let  $g$  be a multiplicative ROP with  $|\text{Var}(g)| \geq 2$ . For every  $x_i \in \text{Var}(g)$ , there exists  $x_j \in \text{Var}(g) \setminus \{x_i\}$  and  $\gamma \in \mathbb{F}$  such that  $\partial_{x_j}(g)|_{x_i=\gamma} = 0$ .*

*Proof.* Let  $\varphi$  be a multiplicative ROF computing  $g$ . As  $|\text{Var}(\varphi)| = |\text{Var}(g)| \geq 2$ ,  $\varphi$  has at least one gate. Let  $v$  be the unique neighbour (parent) of the leaf labeled by  $x_i$ . We denote by  $P_v(\bar{x})$  the ROP that is computed by  $v$ . Assume wlog that  $\text{Var}(P_v) = \{x_1, x_2, \dots, x_{i-1}, x_i\}$ . We show that there exists some ROP  $Q$  such that  $Q(P_v, x_{l+1}, x_{l+2}, \dots, x_n) \equiv f_i$  where  $Q$  and  $P_v$  are variable-disjoint ROPs. Consider the ROF  $\varphi_i$  computing  $f_i$ . Denote with  $\psi_i$  the subformula rooted at  $v$ . The output of  $\psi_i$  is wired as one of the inputs to  $\varphi_i$ . Let the resulting polynomial

computed by  $\varphi_i$  be denoted as  $Q$ . It follows that  $Q(P_v, \bar{x}) \equiv f_i$ . Also  $Q$  and  $P_v$  are variable-disjoint as they are computed by different parts of the same ROP.

Since  $v$  is a multiplication gate (recall that  $\varphi$  is a multiplicative ROF) and neighbor of the leaf labeled by  $x_i$ ,  $P_v$  can be written as  $P_v(\bar{x}) = (x_i - \alpha)h(\bar{x}) + c$  for some ROP  $h$  such that  $\text{Var}(h) \neq \emptyset$  and  $x_i \notin \text{Var}(h)$ .

Finally, by the chain rule, for every variable  $x_j \in \text{Var}(h)$  we have that:

$$\partial_{x_j}(f_i) = \partial_y(Q) \cdot \partial_{x_j}(P_v) = \partial_y(Q) \cdot (x_i - \alpha) \cdot \partial_{x_j}(h)$$

It follows that  $\partial_{x_j}(g) |_{x_i=\alpha} = 0$ .  $\square$

Along with partial derivatives, another operator that we will find useful is the commutator of a polynomial. The commutator of a polynomial has previously been used for polynomial factorization and in reconstruction algorithms for read-once formulas, see [10].

**Definition 3 (Commutator [10]).** Let  $P \in \mathbb{F}[x_1, x_2, \dots, x_n]$  be a multilinear polynomial and let  $i, j \in [n]$ . The commutator between  $x_i$  and  $x_j$ , denoted  $\Delta_{ij}P$ , is defined as follows.

$$\Delta_{ij}P = (P |_{x_i=0, x_j=0}) \cdot (P |_{x_i=1, x_j=1}) - (P |_{x_i=0, x_j=1}) \cdot (P |_{x_i=1, x_j=0})$$

The following property of the commutator will be useful to us.

**Lemma 2.** Let  $f = l_1(x_1, x_2) \cdot l_2(x_3, x_4) + l_3(x_1, x_3) \cdot l_4(x_2, x_4)$  where the  $l_i$ 's are linear polynomials. Then  $l_2$  divides  $\Delta_{12}(f)$ .

*Proof.* First, we show that  $\Delta_{12}(l_3 \cdot l_4) = 0$ . Assume  $l_3 = Cx_1 + m$  and  $l_4 = Dx_2 + n$  where  $C, D \in \mathbb{F}$  and  $m, n$  are linear polynomials in  $x_3, x_4$  respectively. By definition,  $\Delta_{12}(l_3 \cdot l_4) = mn(C + m)(D + n) - m(D + n)(C + m)n = 0$ .

Now we write  $\Delta_{12}f$  explicitly. Let  $l_1 = ax_1 + bx_2 + c$ . By definition,

$$\begin{aligned} \Delta_{12}f &= \Delta_{12}(l_1l_2 + l_3l_4) \\ &= (cl_2 + mn)((a + b + c)l_2 + (C + m)(D + n)) - \\ &\quad ((b + c)l_2 + m(D + n)) \cdot ((a + c)l_2 + n(C + m)) \\ &= l_2^2(c(a + b + c) - (a + c)(b + c)) \\ &\quad + l_2(c(C + m)(D + n) + mn(a + b + c) - n(b + c)(C + m) - m(a + c)(D + n)) \end{aligned}$$

It follows that  $l_2$  divides  $\Delta_{12}f$ .  $\square$

### 3 A proper hierarchy in $\sum^k \cdot \text{ROP}$

This section is devoted to proving Theorem 1.

We prove the lower bound for  $S_n^{n-1}$  by induction. This necessitates a stronger induction hypothesis, so we will actually prove the lower bound for a larger class of polynomials. For any  $\alpha, \beta \in \mathbb{F}$ , we define the polynomial

$\mathcal{M}_n^{\alpha,\beta} = \alpha S_n^n + \beta S_n^{n-1}$ . We note the following recursive structure of  $\mathcal{M}_n^{\alpha,\beta}$ :

$$\begin{aligned} (\mathcal{M}_n^{\alpha,\beta})|_{x_n=\gamma} &= \mathcal{M}_{n-1}^{\alpha\gamma+\beta,\beta\gamma} \\ \partial_{x_n}(\mathcal{M}_n^{\alpha,\beta}) &= \mathcal{M}_{n-1}^{\alpha,\beta} \end{aligned}$$

We show below that each  $\mathcal{M}_n^{\alpha,\beta}$  is expressible as the sum of  $\lceil n/2 \rceil$  ROPs (Lemma 4); however, for any non-zero  $\beta \neq 0$ ,  $\mathcal{M}_n^{\alpha,\beta}$  cannot be written as the sum of fewer than  $\lceil n/2 \rceil$  ROPs (Lemma 3). At  $\alpha = 0, \beta = 1$ , we get  $S_n^{n-1}$ , the simplest such polynomials, establishing Theorem 1.

**Lemma 3.** *Let  $\mathbb{F}$  be a field. For every  $\alpha \in \mathbb{F}$  and  $\beta \in \mathbb{F} \setminus \{0\}$ , the polynomial  $\mathcal{M}_n^{\alpha,\beta} = \alpha S_n^n + \beta S_n^{n-1}$  cannot be written as a sum of  $k < n/2$  ROPs.*

*Proof.* The proof is by induction on  $n$ . The cases  $n = 1, 2$  are easy to see. We now assume that  $k \geq 1$  and  $n > 2k$ . Assume to the contrary that there are ROPs  $f_1, f_2, \dots, f_k$  over  $\mathbb{F}[x_1, x_2, \dots, x_n]$  such that  $\sum_{m \in [k]} f_m = \mathcal{M}_n^{\alpha,\beta}$ . The main steps in the proof are as follows:

1. Show using the inductive hypothesis that for all  $m \in [k]$  and  $a, b \in [n]$ ,  $\partial_{x_a} \partial_{x_b}(f_m) \neq 0$ .
2. Conclude that for all  $m \in [k]$ ,  $f_m$  must be a multiplicative ROP. That is, the ROF computing  $f_m$  does not contain any addition gate.
3. Use the multiplicative property of  $f_k$  to show that  $f_k$  can be eliminated by taking partial derivative with respect to one variable and substituting another by a field constant. If this constant is non-zero, we contradict the inductive hypothesis.
4. Otherwise, use the sum of (multiplicative) ROPs representation of  $\mathcal{M}_n^{\alpha,\beta}$  to show that the degree of  $f$  can be made at most  $(n-2)$  by setting one of the variables to zero. This contradicts our choice of  $f$  since  $\beta \neq 0$ .

We now proceed with the proof.

**Claim 4** *For all  $m \in [k]$  and  $a, b \in [n]$ ,  $\partial_{x_a} \partial_{x_b}(f_m) \neq 0$ .*

*Proof.* Suppose to the contrary that  $\partial_{x_a} \partial_{x_b}(f_m) = 0$ . Assume wlog that  $a = n$ ,  $b = n-1$ ,  $m = k$ , so  $\partial_{x_n} \partial_{x_{n-1}}(f_k) = 0$ . Then,

$$\begin{aligned} \mathcal{M}_n^{\alpha,\beta} &= \sum_{m=0}^k f_m && \text{(by assumption)} \\ \partial_{x_n} \partial_{x_{n-1}}(\mathcal{M}_n^{\alpha,\beta}) &= \sum_{m=0}^k \partial_{x_n} \partial_{x_{n-1}}(f_m) && \text{(by subadditivity of partial derivative)} \\ \mathcal{M}_{n-2}^{\alpha,\beta} &= \sum_{m=0}^{k-1} \partial_{x_n} \partial_{x_{n-1}}(f_m) && \text{(by recursive structure of } \mathcal{M}_n, \\ &&& \text{and since } \partial_{x_n} \partial_{x_{n-1}}(f_k) = 0) \end{aligned}$$

Thus  $\mathcal{M}_{n-2}^{\alpha,\beta}$  can be written as the sum of  $k-1$  polynomials, each of which is an ROP (by Fact 3). By the inductive hypothesis,  $2(k-1) \geq (n-2)$ . Therefore,  $k \geq n/2$  contradicting our assumption.  $\square$

Let  $\varphi_m$  be the ROF computing  $f_m$ . The next step is to show that for  $m \in [k]$ ,  $\varphi_m$  does not contain an addition gate. Suppose to the contrary that for some  $m \in [k]$  and  $p, q \in [n]$ , the least common ancestor of  $x_p$  and  $x_q$  in  $\varphi_m$  is a  $+$  gate. It follows that  $\partial_{x_p} \partial_{x_q} (f_m) = 0$ , contradicting Claim 4. We record this observation below.

**Observation 5** *For all  $m \in [k]$ ,  $f_m$  is a multiplicative ROP.*

Observation 5 and Lemma 1 together imply that for each  $m \in [k]$  and  $i \in [n]$ , there exist  $j \neq i \in [n]$  and  $\gamma \in \mathbb{F}$  such that  $\partial_{x_j} (f_k) |_{x_i=\gamma} = 0$ . There are two cases to consider.

First, consider the case when for some  $m, i$  and the corresponding  $j, \gamma$ , it turns out that  $\gamma \neq 0$ . Assume without loss of generality that  $m = k, i = n-1, j = n$ , so that  $\partial_{x_n} (f_k) |_{x_{n-1}=\gamma} = 0$ . (For other indices the argument is symmetric.) Then

$$\begin{aligned} \mathcal{M}_n^{\alpha,\beta} &= \sum_{i \in [k]} f_i && \text{(by assumption)} \\ \partial_{x_n} (\mathcal{M}_n^{\alpha,\beta}) |_{x_{n-1}=\gamma} &= \sum_{i \in [k]} \partial_{x_n} (f_i) |_{x_{n-1}=\gamma} && \text{(by subadditivity of partial derivative)} \\ \mathcal{M}_{n-1}^{\alpha,\beta} |_{x_{n-1}=\gamma} &= \sum_{i \in [k-1]} \partial_{x_n} (f_i) |_{x_{n-1}=\gamma} && \text{(since } \gamma \text{ is chosen as per Lemma 1)} \\ \mathcal{M}_{n-2}^{\alpha\gamma+\beta, \beta\gamma} &= \sum_{i \in [k-1]} \partial_{x_n} (f_i) |_{x_{n-1}=\gamma} && \text{(recursive structure of } \mathcal{M}_n) \end{aligned}$$

Therefore,  $\mathcal{M}_{n-2}^{\alpha\gamma+\beta, \beta\gamma}$  can be written as a sum of at most  $k-1$  polynomials, each of which is an ROP (Fact 3). By the inductive hypothesis,  $2(k-1) \geq n-2$  implying that  $k \geq n/2$  contradicting our assumption.

(Note: the term  $\mathcal{M}_{n-2}^{\alpha\gamma+\beta, \beta\gamma}$  is what necessitates a stronger induction hypothesis than working with just  $\alpha = 0, \beta = 1$ .)

It remains to handle the case when for all  $m \in [k]$  and  $i \in [n]$ , the corresponding value of  $\gamma$  to some  $x_j$  (as guaranteed by Lemma 1) is 0. Examining the proof of Lemma 1, this implies that each leaf node in any of the ROFs can be made zero only by setting the corresponding variable to zero. That is, the linear forms at all leaves are of the form  $a_i x_i$ .

Since each  $\varphi_m$  is a multiplicative ROP, setting  $x_n = 0$  makes the variables in the polynomial computed at the sibling of the leaf node  $a_n x_n$  redundant. Hence setting  $x_n = 0$  reduces the degree of each  $f_m$  by at least 2. That is,  $\deg(f |_{x_n=0}) \leq n-2$ . But  $f |_{x_n=0}$  equals  $\mathcal{M}_{n-1}^{\beta,0} = \beta S_{n-1}^{n-1}$ , which has degree  $n-1$ , a contradiction.  $\square$

The following lemma shows that the above lower bound is indeed optimal.



**Lemma 4.** *For any field  $\mathbb{F}$  and  $A, B \in \mathbb{F}$ , the polynomial  $f = AS_n^n + BS_n^{n-1}$  can be written as a sum of at most  $\lceil n/2 \rceil$  ROPs.*

*Proof.* Define  $f_i := (x_{2i-1} + x_{2i}) \cdot \left( \prod_{\substack{k \in [n] \\ k \neq 2i, 2i-1}} x_k \right)$ . Notice that each  $f_i$  is a ROP.

Depending on the parity of  $n$ , we consider two cases:

**Case 1:**  $n$  is even;  $n = 2k$ . Then, defining

$$f'_k = (Bx_{2k-1} + Bx_{2k} + Ax_{2k-1}x_{2k}) \cdot \left( \prod_{\substack{m \in [n] \\ m \neq 2k, 2k-1}} x_m \right),$$

we have  $f = B(f_1 + f_2 + \dots + f_{k-1}) + f'_k$ . Note that  $f'_k$  is also an ROP; the factor involving  $x_{2k-1}$  and  $x_{2k}$  is bivariate multilinear and hence an ROP.

**Case 2:**  $n$  is odd;  $n = 2k + 1$ . Then, defining

$$f'_{k+1} = x_1 x_2 \cdots x_{2k} (B + Ax_{2k+1}),$$

we have  $f = B(f_1 + f_2 + \dots + f_k) + f'_{k+1}$ . Note that  $f'_{k+1}$  is also an ROP.

In either case, since all the polynomials  $f_i, f'_k, f'_{k+1}$  are ROPs, we have a representation of  $f$  as a sum of at most  $\lceil n/2 \rceil$  ROPs.  $\square$

Combining the results of Lemma 3 and Lemma 4, we obtain the following theorem. At  $\alpha = 0, \beta = 1$ , it yields Theorem 1.

**Theorem 6.** *For each  $n \geq 1$ , any  $\alpha \in \mathbb{F}$  and any  $\beta \in \mathbb{F} \setminus \{0\}$ , the polynomial  $\alpha S_n^n + \beta S_n^{n-1}$  is in  $\sum^m \cdot \text{ROP}$  but not in  $\sum^{m-1} \cdot \text{ROP}$ , where  $m = \lceil n/2 \rceil$ .*

## 4 A 4-variate multilinear polynomial not in $\sum^2 \cdot \text{ROP}$

This section is devoted to proving Theorem 2. We want to find an explicit 4-variate multilinear polynomial that is not expressible as the sum of 2 ROPs.

Note that the proof of Theorem 1 does not help here, since the polynomials separating  $\sum^2 \cdot \text{ROP}$  from  $\sum^3 \cdot \text{ROP}$  have 5 or 6 variables. One obvious approach is to consider other combinations of the symmetric polynomials. This fails too; we can show that all such combinations are in  $\sum^2 \cdot \text{ROP}$ .

**Proposition 3.** *For every choice of field constants  $a_i$  for each  $i \in \{0, 1, 2, 3, 4\}$ , the polynomial  $\sum_{i=0}^4 a_i S_4^i$  can be expressed as the sum of two ROPs.*

*Proof.* Let  $g = \sum_i a_i S_4^i$ . We obtain the expression for  $g$  in different ways in 4 different cases.

Case	Expression
$a_2 = a_3 = 0$	$g = a_0 + a_1 S_4^1 + a_4 S_4^4$
$a_2 = 0;$ $a_3 \neq 0$	$g = \left( a_1 + a_3 x_1 x_2 \right) \left( x_3 + x_4 + \frac{a_4}{a_3} x_3 x_4 \right) + \left( (a_1 + a_3 x_3 x_4) \left( x_1 + x_2 - \frac{a_1 a_4}{a_3^2} \right) \right) + c$
$a_2 \neq 0;$ $a_2 a_4 = a_3^2$	$a_2 g = (a_1 + a_2(x_1 + x_2) + a_3 x_1 x_2)(a_1 + a_2(x_3 + x_4) + a_3 x_3 x_4) + (a_2^2 - a_1 a_3)(x_1 x_2 + x_3 x_4) + c$
$a_2 \neq 0;$ $a_2 a_4 \neq a_3^2$	$a_2 g = (a_1 + a_2(x_1 + x_2) + a_3 x_1 x_2)(a_1 + a_2(x_3 + x_4) + a_3 x_3 x_4) + \left( x_1 x_2 + \frac{a_2^2 - a_1 a_3}{a_2 a_4 - a_3^2} \right) ((a_2 a_4 - a_3^2) x_3 x_4 + a_2^2 - a_1 a_3) + c$

In the above,  $c$  is an appropriate field constant, and can be added to any ROP. Notice that the first expression is a sum of two ROPs since it is the sum of a linear polynomial and a single monomial. All the other expressions have two summands, each of which is a product of variable-disjoint bivariate polynomials (ignoring constant terms). Since every bivariate polynomial is a ROP, these representations are also sums of 2 ROPs.  $\square$

Instead, we define a polynomial that gives carefully chosen weights to the monomials of  $S_4^2$ . Let  $f^{\alpha, \beta, \gamma}$  denote the following polynomial:

$$f^{\alpha, \beta, \gamma} = \alpha \cdot (x_1 x_2 + x_3 x_4) + \beta \cdot (x_1 x_3 + x_2 x_4) + \gamma \cdot (x_1 x_4 + x_2 x_3).$$

To keep notation simple, we will omit the superscript when it is clear from the context. In the theorem below, we obtain necessary and sufficient conditions on  $\alpha, \beta, \gamma$  under which  $f$  can be expressed as a sum of two ROPs.

**Theorem 7 (Hardness of representation for sum of 2 ROPs).** *Let  $f$  be the polynomial  $f^{\alpha, \beta, \gamma} = \alpha \cdot (x_1 x_2 + x_3 x_4) + \beta \cdot (x_1 x_3 + x_2 x_4) + \gamma \cdot (x_1 x_4 + x_2 x_3)$ . The following are equivalent:*

1.  $f$  is not expressible as the sum of two ROPs.
2.  $\alpha, \beta, \gamma$  satisfy all the three conditions C1, C2, C3 listed below.
  - C1:**  $\alpha \beta \gamma \neq 0$ .
  - C2:**  $(\alpha^2 - \beta^2)(\beta^2 - \gamma^2)(\gamma^2 - \alpha^2) \neq 0$ .
  - C3:** None of the equations  $X^2 - D_i = 0$ ,  $i \in [3]$ , has a root in  $\mathbb{F}$ , where

$$\begin{aligned} D_1 &= (+\alpha^2 - \beta^2 - \gamma^2)^2 - (2\beta\gamma)^2 \\ D_2 &= (-\alpha^2 + \beta^2 - \gamma^2)^2 - (2\alpha\gamma)^2 \\ D_3 &= (-\alpha^2 - \beta^2 + \gamma^2)^2 - (2\alpha\beta)^2 \end{aligned}$$

*Remark 1.* 1. It follows, for instance, that  $2(x_1 x_2 + x_3 x_4) + 4(x_1 x_3 + x_2 x_4) + 5(x_1 x_4 + x_2 x_3)$  cannot be written as a sum of 2 ROPs over reals, yielding Theorem 2.

2. If  $\mathbb{F}$  is an algebraically closed field, then for every  $\alpha, \beta, \gamma$ , condition C3 fails, and so every  $f^{\alpha, \beta, \gamma}$  can be written as a sum of 2 ROPs. However we do not know if there are other examples, or whether all multilinear 4-variate polynomials are expressible as the sum of two ROPs.
3. Even if  $\mathbb{F}$  is not algebraically closed, condition C3 fails if for each  $a \in \mathbb{F}$ , the equation  $X^2 = a$  has a root.

To prove Theorem 7, we first consider the easier direction,  $1 \Rightarrow 2$ , and prove the contrapositive.

**Lemma 5.** *If  $\alpha, \beta, \gamma$  do not satisfy all of C1, C2, C3, then the polynomial  $f$  can be written as a sum of 2 ROPs.*

*Proof.* **C1 false:** If any of  $\alpha, \beta, \gamma$  is zero, then by definition  $f$  is the the sum of at most two ROPs.

**C2 false:** Without loss of generality, assume  $\alpha^2 = \beta^2$ , so  $\alpha = \pm\beta$ . Then  $f$  is computed by  $f = \alpha \cdot (x_1 \pm x_4)(x_2 \pm x_3) + \gamma \cdot (x_1x_4 + x_2x_3)$ .

**C1 true; C3 false:** Without loss of generality, the equation  $X^2 - D_1 = 0$  has a root  $\tau$ . We try to express  $f$  as

$$\alpha(x_1 - Ax_3)(x_2 - Bx_4) + \beta(x_1 - Cx_2)(x_3 - Dx_4).$$

The coefficients for  $x_3x_4$  and  $x_2x_4$  force  $AB = 1$ ,  $CD = 1$ , giving the form

$$\alpha(x_1 - Ax_3)(x_2 - \frac{1}{A}x_4) + \beta(x_1 - Cx_2)(x_3 - \frac{1}{C}x_4).$$

Comparing the coefficients for  $x_1x_4$  and  $x_2x_3$ , we obtain the constraints

$$-\frac{\alpha}{A} - \frac{\beta}{C} = \gamma; \quad -\alpha A - \beta C = \gamma$$

Expressing  $A$  as  $\frac{-\gamma - \beta C}{\alpha}$ , we get a quadratic constraint on  $C$ ; it must be a root of the equation

$$Z^2 + \frac{-\alpha^2 + \beta^2 + \gamma^2}{\beta\gamma}Z + 1 = 0.$$

Using the fact that  $\tau^2 = D_1 = (-\alpha^2 + \beta^2 + \gamma^2)^2 - (2\beta\gamma)^2$ , we see that indeed this equation does have roots. The left-hand side splits into linear factors, giving

$$(Z - \delta)(Z - \frac{1}{\delta}) = 0 \quad \text{where} \quad \delta = \frac{\alpha^2 - \beta^2 - \gamma^2 + \tau}{2\beta\gamma}.$$

It is easy to verify that  $\delta \neq 0$  and  $\delta \neq -\frac{\gamma}{\beta}$  (since  $\alpha \neq 0$ ). Further, define  $\mu = \frac{-(\gamma + \beta\delta)}{\alpha}$ . Then  $\mu$  is well-defined (because  $\alpha \neq 0$ ) and is also non-zero. Now setting  $\tilde{C} = \delta$  and  $A = \mu$ , we have satisfied all the constraints and so we can write  $f$  as the sum of 2 ROPs as follows:

$$f = \alpha(x_1 - \mu x_3)(x_2 - \frac{1}{\mu}x_4) + \beta(x_1 - \delta x_2)(x_3 - \frac{1}{\delta}x_4).$$

□

Now we consider the harder direction:  $2 \Rightarrow 1$ . Again, we consider the contrapositive. We first show (Lemma 6) a structural property satisfied by every polynomial in  $\sum^2 \cdot \text{ROP}$ : it must satisfy at least one of the three properties  $C1', C2', C3'$  described in the lemma. We then show (Lemma 7) that under the conditions  $C1, C2, C3$  from the theorem statement,  $f$  does not satisfy any of  $C1', C2', C3'$ ; it follows that  $f$  is not expressible as the sum of 2 ROPs.

**Lemma 6.** *Let  $g$  be a 4-variate multilinear polynomial over the field  $\mathbb{F}$  which can be expressed as a sum of 2 ROPs. Then at least one of the following conditions is true:*

- C1':** *There exist  $i, j \in [4]$  and  $A, B \in \mathbb{F}$  such that  $g|_{x_i=A, x_j=B}$  is linear.*
- C2':** *There exist  $i, j \in [4]$  such that  $x_i, x_j, \partial_{x_i}(g), \partial_{x_j}(g)$  are affinely dependent.*
- C3':**  *$g = l_1 \cdot l_2 + l_3 \cdot l_4$  where  $l_i$ s are variable disjoint linear forms.*

*Proof.* Let  $\varphi$  be a sum of 2 ROPs computing  $g$ . Let  $v_1$  and  $v_2$  be the children of the topmost  $+$  gate. The proof is in two steps. First, we reduce to the case when  $|\text{Var}(v_1)| = |\text{Var}(v_2)| = 4$ . Then we use a case analysis to show that at least one of the aforementioned conditions hold true. In both steps, we will repeatedly use Proposition 2, which showed that any 3-variate ROP can be reduced to a linear polynomial by substituting a single variable with a field constant. We now proceed with the proof.

Suppose  $|\text{Var}(v_1)| \leq 3$ . Applying Proposition 2 first to  $v_1$  and then to the resulting restriction of  $v_2$ , one can see that there exist  $i, j \in [4]$  and  $A, B \in \mathbb{F}$  such that  $g|_{x_i=A, x_j=B}$  is a linear polynomial. So condition  $C1'$  is satisfied.

Now assume that  $|\text{Var}(v_1)| = |\text{Var}(v_2)| = 4$ . Depending on the type of gates of  $v_1$  and  $v_2$ , we consider 3 cases.

**Case 1:** Both  $v_1$  and  $v_2$  are  $\times$  gates. Then  $g$  can be represented as  $M_1 \cdot M_2 + M_3 \cdot M_4$  where  $(M_1, M_2)$  and  $(M_3, M_4)$  are variable-disjoint ROPs.

Suppose that for some  $i$ ,  $|\text{Var}(M_i)| = 1$ . Then,  $g|_{M_i \rightarrow 0}$  is a 3-variate restriction of  $f$  and is clearly an ROP. Applying Proposition 2 to this restriction, we see that condition  $C1'$  holds.

Otherwise each  $M_i$  has  $|\text{Var}(M_i)| = 2$ .

Suppose  $(M_1, M_2)$  and  $(M_3, M_4)$  define distinct partitions of the variable set. Assume wlog that  $g = M_1(x_1, x_2) \cdot M_2(x_3, x_4) + M_3(x_1, x_3) \cdot M_4(x_2, x_4)$ . If all  $M_i$ s are linear forms, it is clear that condition  $C3'$  holds. If not, assume that  $M_1$  is of the form  $l_1(x_1) \cdot m_1(x_2) + c_1$  where  $l_1, m_1$  are linear forms and  $c_1 \in \mathbb{F}$ . Now  $g|_{l_1 \rightarrow 0} = c_1 \cdot M_2(x_3, x_4) + M'_3(x_3) \cdot M_4(x_2, x_4)$ . Either set  $x_3$  to make  $M'_3$  zero, or, if that is not possible because  $M'_3$  is a non-zero field constant, then set  $x_4 \rightarrow B$  where  $B \in \mathbb{F}$ . In both cases, by setting at most 2 variables, we obtain a linear polynomial, so  $C1'$  holds.

Otherwise,  $(M_1, M_2)$  and  $(M_3, M_4)$  define the same partition of the variable set. Assume wlog that  $g = M_1(x_1, x_2) \cdot M_2(x_3, x_4) + M_3(x_1, x_2) \cdot M_4(x_3, x_4)$ . If one of the  $M_i$ s is linear, say wlog that  $M_1$  is a linear form, then  $g|_{M_4 \rightarrow 0}$  is a 2-variate restriction which is also a linear form, so  $C1'$  holds. Otherwise, none of the  $M_i$ s is a linear form. Then each  $M_i$  can be represented as  $l_i \cdot m_i + c_i$  where

$l_i, m_i$  are univariate linear forms and  $c_i \in \mathbb{F}$ . We consider a 2-variate restriction which sets  $l_1$  and  $m_4$  to 0. (Note that  $\text{Var}(l_1) \cap \text{Var}(m_4) = \emptyset$ .) Then the resulting polynomial is a linear form, so  $C1'$  holds.

**Case 2:** Both  $v_1$  and  $v_2$  are  $+$  gates. Then  $g$  can be written as  $f = M_1 + M_2 + M_3 + M_4$  where  $(M_1, M_2)$  and  $(M_3, M_4)$  are variable-disjoint ROPs.

Suppose  $(M_1, M_2)$  and  $(M_3, M_4)$  define distinct partitions of the variable set.

Suppose further that there exists  $M_i$  such that  $|\text{Var}(M_i)| = 1$ . Wlog,  $\text{Var}(M_1) = \{x_1\}$ ,  $\{x_1, x_2\} \subseteq \text{Var}(M_3)$ , and  $x_3 \in \text{Var}(M_4)$ . Any setting to  $x_2$  and  $x_4$  results in a linear polynomial, so  $C1'$  holds.

So assume wlog that  $g = M_1(x_1, x_2) + M_2(x_3, x_4) + M_3(x_1, x_3) + M_4(x_2, x_4)$ . Then for  $A, B \in \mathbb{F}$ ,  $g|_{x_1=A, x_4=B}$  is a linear polynomial, so  $C1'$  holds.

Otherwise,  $(M_1, M_2)$  and  $(M_3, M_4)$  define the same partition of the variable set. Again, if say  $|\text{Var}(M_1)| = 1$ , then setting two variables from  $M_2$  shows that  $C1'$  holds. So assume wlog that  $g = M_1(x_1, x_2) + M_2(x_3, x_4) + M_3(x_1, x_2) + M_4(x_3, x_4)$ . Then for  $A, B \in \mathbb{F}$ ,  $g|_{x_1=A, x_3=B}$  is a linear polynomial, so again  $C1'$  holds.

**Case 3:** One of  $v_1, v_2$  is a  $+$  gate and the other is a  $\times$  gate. Then  $g$  can be written as  $g = M_1 + M_2 + M_3 \cdot M_4$  where  $(M_1, M_2)$  and  $(M_3, M_4)$  are variable-disjoint ROPs. Suppose that  $|\text{Var}(M_3)| = 1$ . Then  $g|_{M_3 \rightarrow 0}$  is a 3-variate restriction which is a ROP. Using Proposition 2, we get a 2-variate restriction of  $g$  which is also linear, so  $C1'$  holds. The same argument works when  $|\text{Var}(M_4)| = 1$ . So assume that  $M_3$  and  $M_4$  are bivariate polynomials.

Suppose that  $(M_1, M_2)$  and  $(M_3, M_4)$  define distinct partitions of the variable set. Assume wlog that  $g = M_1 + M_2 + M_3(x_1, x_2) \cdot M_4(x_3, x_4)$ , and  $x_3, x_4$  are separated by  $M_1, M_2$ . Then  $g|_{M_3 \rightarrow 0}$  is a 2-variate restriction which is also linear, so  $C1'$  holds.

Otherwise  $(M_1, M_2)$  and  $(M_3, M_4)$  define the same partition of the variable set. Assume wlog that  $g = M_1(x_1, x_2) + M_2(x_3, x_4) + M_3(x_1, x_2) \cdot M_4(x_3, x_4)$ . If  $M_1$  (or  $M_2$ ) is a linear form, then consider a 2-variate restriction of  $g$  which sets  $M_4$  (or  $M_3$ ) to 0. The resulting polynomial is a linear form. Similarly if  $M_3$  (or  $M_4$ ) is of the form  $l \cdot m + c$  where  $l, m$  are univariate linear forms, then we consider a 2-variate restriction which sets  $l$  to 0 and some  $x_i \in \text{Var}(M_4)$  to a field constant. The resulting polynomial again is a linear form. In all these cases,  $C1'$  holds.

The only case that remains is that  $M_3$  and  $M_4$  are linear forms while  $M_1$  and  $M_2$  are not. Assume that  $M_1 = (A_1x_1 + B_1)(A_2x_2 + B_2) + C$  and  $M_3 = A_3x_1 + B_3x_2 + C_3$ . Then  $\partial_{x_1}(g) = A_1(A_2x_2 + B_2) + A_3M_4$  and  $\partial_{x_2}(g) = (A_1x_1 + B_1)A_2 + B_3M_4$ . It follows that  $B_3 \cdot \partial_{x_1}(g) - A_3 \cdot \partial_{x_2}(g) + A_1A_2A_3x_1 - A_1A_2B_3x_2 \in \mathbb{F}$ , and hence the polynomials  $x_1, x_2, \partial_{x_1}(g), \partial_{x_2}(g)$  are affinely dependent. Therefore, condition  $C2'$  of the lemma is satisfied.  $\square$

**Lemma 7.** *If  $\alpha, \beta, \gamma$  satisfy conditions  $C1, C2, C3$  from the statement of Theorem 7, then the polynomial  $f^{\alpha, \beta, \gamma}$  does not satisfy any of the properties  $C1', C2', C3'$  from Lemma 6.*

*Proof.* **C1**  $\Rightarrow$   $\neg$ **C1'**: Since  $\alpha\beta\gamma \neq 0$ ,  $f$  contains all possible degree 2 monomials. Hence after setting  $x_i = A$  and  $x_j = B$ , the monomial  $x_k x_l$  where  $k, l \in [4] \setminus \{i, j\}$  still survives.

**C2**  $\Rightarrow$   $\neg$ **C2'**: The proof is by contradiction. Assume to the contrary that for some  $i, j$ , wlog say for  $i = 1$  and  $j = 2$ , the polynomials  $x_1, x_2, \partial_{x_1}(f), \partial_{x_2}(f)$  are affinely dependent. Note that  $\partial_{x_1}(f) = \alpha x_2 + \beta x_3 + \gamma x_4$  and  $\partial_{x_2}(f) = \alpha x_1 + \gamma x_3 + \beta x_4$ . This implies that the vectors  $(1, 0, 0, 0), (0, 1, 0, 0), (0, \alpha, \beta, \gamma)$  and  $(\alpha, 0, \gamma, \beta)$  are linearly dependent. This further implies that the vectors  $(\beta, \gamma)$  and  $(\gamma, \beta)$  are linearly dependent. Therefore,  $\beta = \pm\gamma$ , contradicting **C2**.

**C1**  $\wedge$  **C2**  $\wedge$  **C3**  $\Rightarrow$   $\neg$ **C3'**: Suppose, to the contrary, that **C3'** holds. That is,  $f$  can be written as  $f = l_1 \cdot l_2 + l_3 \cdot l_4$  where  $(l_1, l_2)$  and  $(l_3, l_4)$  are variable-disjoint linear forms. By the preceding arguments, we know that  $f$  does not satisfy **C1'** or **C2'**.

First consider the case when  $(l_1, l_2)$  and  $(l_3, l_4)$  define the same partition of the variable set. Assume wlog that  $\text{Var}(l_1) = \text{Var}(l_3)$ ,  $\text{Var}(l_2) = \text{Var}(l_4)$ , and  $|\text{Var}(l_1)| \leq 2$ . Setting the variables in  $l_1$  to any field constants yields a linear form, so  $f$  satisfies **C1'**, a contradiction.

Hence it must be the case that  $(l_1, l_2)$  and  $(l_3, l_4)$  define different partitions of the variable set. Since all degree-2 monomials are present in  $f$ , each pair  $x_i, x_j$  must be separated by at least one of the two partitions. This implies that both partitions have exactly 2 variables in each part. Assume without loss of generality that  $f = l_1(x_1, x_2) \cdot l_2(x_3, x_4) + l_3(x_1, x_3) \cdot l_4(x_2, x_4)$ .

At this point, we use properties of the commutator of  $f$ ; recall Definition 3. By Lemma 2, we know that  $l_2$  divides  $\Delta_{12}f$ . We compute  $\Delta_{12}f$  explicitly for our candidate polynomial:

$$\begin{aligned}\Delta_{12}f &= (\alpha x_3 x_4)(\alpha + (\beta + \gamma)(x_3 + x_4) + \alpha x_3 x_4) \\ &\quad - (\beta x_4 + \gamma x_3 + \alpha x_3 x_4)(\beta x_3 + \gamma x_4 + \alpha x_3 x_4) \\ &= -\beta\gamma(x_3^2 + x_4^2) + (\alpha^2 - \beta^2 - \gamma^2)x_3 x_4\end{aligned}$$

Since  $l_2$  divides  $\Delta_{12}f$ ,  $\Delta_{12}f$  is not irreducible but is the product of two linear factors. Since  $\Delta_{12}f(0, 0) = 0$ , at least one of the linear factors of  $\Delta_{12}f$  must vanish at  $(0, 0)$ . Let  $x_3 - \delta x_4$  be such a factor. Then  $\Delta_{12}(f)$  vanishes not only at  $(0, 0)$ , but whenever  $x_3 = \delta x_4$ . Substituting  $x_3 = \delta x_4$  in  $\Delta_{12}f$ , we get

$$-\delta^2\beta\gamma - \beta\gamma + \delta(\alpha^2 - \beta^2 - \gamma^2) = 0$$

Hence  $\delta$  is of the form

$$\delta = \frac{-(\alpha^2 - \beta^2 - \gamma^2) \pm \sqrt{(\alpha^2 - \beta^2 - \gamma^2)^2 - 4\beta^2\gamma^2}}{-2\beta\gamma}$$

Hence  $2\beta\gamma\delta - (\alpha^2 - \beta^2 - \gamma^2)$  is a root of the equation  $X^2 - D_1 = 0$ , contradicting the assumption that **C3** holds.

Hence it must be the case that **C3'** does not hold.  $\square$

With this, the proof of Theorem 7 is complete.

The conditions imposed on  $\alpha, \beta, \gamma$  in Theorem 7 are tight and irredundant. Below we give some explicit examples over the field of reals.

1.  $f = 2(x_1x_2 + x_3x_4) + 2(x_1x_3 + x_2x_4) + 3(x_1x_4 + x_2x_3)$  satisfies conditions C1 and C3 from the Theorem but not C2;  $\alpha = \beta$ . A  $\sum^2$ -ROP representation for  $f$  is  $f = 2(x_1 + x_4)(x_2 + x_3) + 3(x_1x_4 + x_2x_3)$ .
2.  $f = 2(x_1x_2 + x_3x_4) - 2(x_1x_3 + x_2x_4) + 3(x_1x_4 + x_2x_3)$  satisfies conditions C1 and C3 but not C2;  $\alpha = -\beta$ . A  $\sum^2$ -ROP representation for  $f$  is  $f = 2(x_1 - x_4)(x_2 - x_3) + 3(x_1x_4 + x_2x_3)$ .
3.  $f = (x_1x_2 + x_3x_4) + 2(x_1x_3 + x_2x_4) + 3(x_1x_4 + x_2x_3)$  satisfies conditions C1 and C2 but not C3. A  $\sum^2$ -ROP representation for  $f$  is  $f = (x_1 + x_3)(x_2 + x_4) + 2(x_1 + x_2)(x_3 + x_4)$ .

## 5 Conclusions

1. We have seen in Proposition 1 that every  $n$ -variate multilinear polynomial ( $n \geq 4$ ) can be written as the sum of  $3 \times 2^{n-4}$  ROPs. We have also shown in Lemma 3 that there are  $n$ -variate multilinear polynomials that require  $\lceil n/2 \rceil$  ROPs in a sum-of-ROPs representation. Between  $\lceil n/2 \rceil$  and  $3 \times 2^{n-4}$ , what is the true tight bound?
2. We have shown in Theorem 1 that for each  $k$ ,  $\sum^k$ -ROP can be separated from  $\sum^{k-1}$ -ROP by a polynomial on  $2k - 1$  variables. Can we separate these classes with fewer variables? Note that any separating polynomial must have  $\Omega(\log k)$  variables.
3. In particular, can 4-variate multilinear polynomials separate sums of 3 ROPs from sums of 2 ROPs over every field? If not, what is an explicit example?

## References

1. Daoud Bshouty and Nader H. Bshouty. On interpolating arithmetic read-once formulas with exponentiation. *J. Comput. Syst. Sci.*, 56(1):112–124, 1998.
2. Nader H. Bshouty and Richard Cleve. Interpolating arithmetic read-once formulas in parallel. *SIAM J. Comput.*, 27(2):401–413, 1998.
3. Nader H. Bshouty, Thomas R. Hancock, and Lisa Hellerstein. Learning boolean read-once formulas over generalized bases. *J. Comput. Syst. Sci.*, 50(3):521–542, 1995.
4. Thomas R. Hancock and Lisa Hellerstein. Learning read-once formulas over fields and extended bases. In Manfred K. Warmuth and Leslie G. Valiant, editors, *Proceedings of the Fourth Annual Workshop on Computational Learning Theory, COLT 1991, Santa Cruz, California, USA, August 5-7, 1991*, pages 326–336. Morgan Kaufmann, 1991.
5. Valentine Kabanets and Russell Impagliazzo. Derandomizing polynomial identity tests means proving circuit lower bounds. *Computational Complexity*, 13(1-2):1–46, 2004.

6. Neeraj Kayal, Pascal Koiran, Timothée Pecatte, and Chandan Saha. Lower bounds for sums of powers of low degree univariates. In Magnús M. Halldórsson, Kazuo Iwama, Naoki Kobayashi, and Bettina Speckmann, editors, *Automata, Languages, and Programming - 42nd International Colloquium, ICALP 2015, Kyoto, Japan, July 6-10, 2015, Proceedings, Part I*, volume 9134 of *Lecture Notes in Computer Science*, pages 810–821. Springer, 2015.
7. Ran Raz. Multi-linear formulas for permanent and determinant are of super-polynomial size. *J. ACM*, 56(2), 2009.
8. Amir Shpilka and Ilya Volkovich. Improved polynomial identity testing for read-once formulas. In Irit Dinur, Klaus Jansen, Joseph Naor, and José D. P. Rolim, editors, *Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques, 12th International Workshop, APPROX 2009, and 13th International Workshop, RANDOM 2009, Berkeley, CA, USA, August 21-23, 2009. Proceedings*, volume 5687 of *Lecture Notes in Computer Science*, pages 700–713. Springer, 2009.
9. Amir Shpilka and Ilya Volkovich. Read-once polynomial identity testing. *Electronic Colloquium on Computational Complexity (ECCC)*, 17:11, 2010.
10. Amir Shpilka and Ilya Volkovich. On reconstruction and testing of read-once formulas. *Theory of Computing*, 10:465–514, 2014.